Letters 29A, 88 (1969). Later measurements [G. Bonera, F. Borsa, and A. Rigamonti (unpublished)] of Na^{23} quadrupole relaxation in deuterated Rochelle Salt (kindly supplied by J. L. Bjorkstam) have confirmed the interpretation given in the Phys. Letters work regarding the frequency dependence of the rapid rise of the relaxation rate attributed to the flipping motion of the water molecules.

[11] R. Blinc, S. Zumer and G. Lahajnar, Phys. Rev. B (to be published).

[12] J. Van Kranendonk, Physica 20, 781 (1954).

[13] A. Abragam, *Principles of Nuclear Magnetic* (Oxford U. P., Oxford, 1961).

[14] An evaluation of the second-order contribution in the case of damped phonons is under way by H. Pytte and X. Von der Wardkirch.

[15] See, for example, E. G. Wikner, W. E. Blumberg, and E. L. Hahn, Phys. Rev. 118, 631 (1960); E. G. Weber, *ibid.* 130, 1 (1963); S. K. Joshi, R. Gupta, and T. P. Das, *ibid.* 134, 1693 (1964); F. Bridges and W. G. Clark, *ibid.* 164, 288 (1967).

[16] B. I. Kochelaev, Zh. Eksperim i Teor. Fiz. 37, 242 (1959) [Soviet Phy. JETP 10, 171 (1960)].

[17] Recently, P. Papon and H. Theveneau [Phys. Letters 30A, 362 (1969)] have calculated the relaxation rate near T_c for an anharmonic Raman process. They find a singular behavior as $(T-T_0)^{-3/2}$ by using a dispersion curve of the type $\omega^2=\omega_0^2+aq^2$.

[18] It should be noted that in any case the succeeding terms in the q expansion of $A_\mu(\vec{q})$ do not give singular contributions to the relaxation rate.

[19] M. Suzuki and R. Kubo, J. Phys. Soc. Japan 24, 51 (1968).

[20] K. Tani, J. Phys. Soc. Japan 26, 93 (1969).

[21] E. Pytte and H. Thomas, Phys. Rev. 175, 610 (1968).

[22] The single crystal of sodium nitrite was kindly provided by A. Weiss.

[23] Equations (30) and (31) refer to the case of an axially symmetric efg tensor. For the general case, see M. H. Cohen and F. Reif, in *Solid State Physics*, edited by F. Seitz and D. Turnbull (Academic, New York, 1957), Vol. 5.

[24] For the features of the transition, see S. Tanisaki, J. Phys. Soc. Japan 18, 1181 (1963) and references therein.

[25] It should be pointed out that a para-antiferroelectric transition was reported to occur immediately before the ferroelectric transition [Y. Yamada, I. Shibuya, and S. Hoshino, J. Phys. Soc. Japan 18, 1594 (1963)]. However, the effect of this transition on the dielectric constant is small [K. Gesi, J. Appl. Phys. (Japan) 4, 818 (1965)] and it does not show up in our measurements.

[26] A. Weiss, Z. Naturforsch. 15A, 536 (1960); A. Weiss and D. Biedenkapp, *ibid.* 17A, 794 (1962).

[27] H. Betsuyaku, J. Phys. Soc. Japan 21, 187 (1966).

[28] K. Gesi, J. Phys. Soc. Japan 26, 953 (1969).

[29] I. Hatta, J. Phys. Soc. Japan 24, 1043 (1968); Y. Yamada, Y. Fujii, and I. Hatta, *ibid.* 24, 1053 (1968).

[30] M. K. Barnoski and J. M. Ballantyne, Phys. Rev. 174, 946 (1968).

[31] H. Betsuyaku, J. Chem. Phys. 51, 2546 (1969).

PHYSICAL REVIEW B VOLUME 2, NUMBER 7 1 OCTOBER 1970

Theory of a Two-Dimensional Ising Model with Random Impurities. IV. Generalizations

Barry M. McCoy
Institute for Theoretical Physics, State University of New York, Stony Brook, New York* 11790
(Received 1 June 1970)

In the first paper of this series, we located T_c for an arbitrary distribution of impurity bonds $P(E_2)$ and, for a particular $P(E_2)$ with a narrow width, we found that the specific heat fails to be analytic at T_c, although it is infinitely differentiable there. In this paper, we generalize this conclusion to an arbitrary distribution $P(E_2)$.

I. INTRODUCTION

In the first paper of this series,[1] we introduced a two-dimensional Ising model in which the horizontal energies E_1 are all fixed and all the vertical bonds $E_2(j)$ between the jth row and the $(j+1)$th row are fixed, but $E_2(j)$ is allowed to depend on j. Furthermore, the energies $E_2(j)$ are treated as random variables such that if $j\neq j'$, $E_2(j)$ and $E_2(j')$ are independent and are each described by a temperature-independent probability density $P(E_2)$. For this Ising model, we showed that with probability 1 the thermodynamic limit of the free energy per site exists and is the same for all Ising models of the collection specified by $P(E_2)$.

For this random Ising lattice we located T_c by the general formula

$$0=\int_0^\infty dE_2\,P(E_2)\,\ln\left[z_{2c}^2\left(\frac{1+z_{1c}}{1-z_{1c}}\right)^2\right], \qquad (1.1)$$

where

$$z_i=\tanh\beta E_i\ , \quad i=1,2 \qquad (1.2)$$

and the subscript c means $T=T_c$. To justify calling the T_c located from (1.1) a critical temperature, we would like to show that the specific heat

C_H at $H=0$ fails to be an analytic function of T at T_c. In I we made no such statement for a general distribution function $P(E_2)$. Instead, we considered the special case

$$\mu(\lambda) = N\lambda_0^{-N}\lambda^{N-1} \quad \text{for } 0 \le \lambda \le \lambda_0 \qquad = 0 \quad \text{otherwise} \tag{1.3}$$

where

$$\lambda = z_2^2 \tag{1.4}$$

and

$$P(E_2)dE_2 = \mu(\lambda)d\lambda , \tag{1.5}$$

and showed that the terms in $C_{H=0}$ that are of order 1 as $N \to \infty$ are analytic except at the T_c defined by (1.1). Moreover, we showed that at this T_c these terms in $C_{H=0}$, even though they fail to be analytic, are infinitely differentiable functions of the real variable T.

The purpose of this paper is to generalize the conclusion of I from the special case (1.3) to the general case of an arbitrary probability distribution $P(E_2)$. More specifically, we show that for any $P(E_2)$, $C_{H=0}$ fails to be an analytic function of T at the T_c defined by (1.1), and that at T_c the form of the singularity is the same as that found in I for the special case (1.3) unless $P(E_2) = \delta(E_2 - E_2^0)$. We also show that if

$$P(E_2) \neq 0 \quad \text{only if } E_2' < E_2 < E_2^0 , \tag{1.6}$$

then $C_{H=0}$ is an analytic function of T if T is low enough so that

$$(1-z_1)/(1+z_1) < z_2' \tag{1.7a}$$

or high enough so that

$$z_2^0 < (1-z_1)/(1+z_1). \tag{1.7b}$$

However, we have not been able to prove the stronger statement that, if T is such that (1.7) does not hold, then $C_{H=0}$ fails to be analytic *only* at T_c.

II. FORMULATION OF PROBLEM

In I we showed that the free energy of the random lattice is given by

$$F_r = -\beta\Big[\ln(\cosh\beta E_1) + \int_0^\infty dE_2\, P(E_2)\ln\cosh\beta E_2 + (4\pi)^{-1}\int_{-\pi}^{\pi} d\theta\, \ln\left|1+z_1 e^{i\theta}\right|^2 + (2\pi)^{-1}\int_{-\pi}^{0} d\theta \times \int_0^\infty dx\,\nu(x)\int_0^\infty dE_2\, P(E_2)\ln(a^2+b^2+az_2^2x^{-1})\Big] , \tag{2.1}$$

where

$$a = -2\, z_1 \sin\theta\, \left|1+z_1 e^{i\theta}\right|^{-2} \tag{2.2}$$

and

$$b = (1-z_1^2)\left|1+z_1 e^{i\theta}\right|^{-2} . \tag{2.3}$$

Furthermore, in (2.1) the function $\nu(x)$ is zero unless

$$ax_0(1) < ax < a^2+b^2 , \tag{2.4}$$

where

$$x_0(\lambda) = (2a)^{-1}\{a^2+b^2-\lambda+[(a^2+b^2-\lambda)^2+4\lambda a^2]^{1/2}\} , \tag{2.5}$$

with

$$x_0(\lambda)^{-1} = (2a\lambda)^{-1}\{-(a^2+b^2-\lambda) + [(a^2+b^2-\lambda)^2+4\lambda a^2]^{1/2}\} , \tag{2.6}$$

and, if (2.4) holds (and $a>0$), $\nu(x)$ satisfies the integral equation

$$\nu(x) = \frac{b^2}{(x-a)^2}\int_{x_0(1)}^{\min[(x-a)/(a^2+b^2-ax),(a^2+b^2)/a]} dx' \times x'\,\nu(x')\mu\left(x'\,\frac{a^2+b^2-ax}{x-a}\right) . \tag{2.7}$$

In this paper we show that for any $P(E_2)$, F_r fails to be analytic at $T=T_c$, where T_c is given by (1.1). By definition, $P(E_2)$ is independent of T. Therefore, the first three terms in (2.1) are analytic functions of T for real T. To study the last term in (2.1), it is convenient to break the θ integration into two parts as shown in Eqs. (2.8) and (2.9):

$$-\pi \le \theta \le \epsilon < 0 . \tag{2.8}$$

In the case of Eq. (2.8) the resulting expression will be shown to be an analytic function of T when T is real and non-negative,

$$\epsilon \le \theta \le 0 . \tag{2.9}$$

This region will be shown to give rise to an infinitely differentiable singularity at T_c of precisely the same nature as that found in I.

III. $-\pi \le \theta \le \epsilon < 0$

We consider the last term in (2.1) with θ restricted by (2.8):

$$G(T) = \int_{-\pi}^{\epsilon} d\theta \int_0^\infty dx\,\nu(x) \times \int_0^\infty dE_2\, P(E_2)\ln(a^2+b^2+az^2/x), \tag{3.1a}$$

which, by (2.22) of *I*, may be reexpressed as

$$G(T) = \int_{-\pi}^{\epsilon} d\theta \int_0^\infty dx\,\nu(x)\int_0^\infty dE_2\, P(E_2)\ln(z_2^2+ax) . \tag{3.1b}$$

Unfortunately, (2.1) is valid only for real T and hence is not particularly suitable for investigating analyticity properties as a function of T. Therefore, to proceed further, we generalize (2. 1) and (2. 7) to the case that T is complex and lies in some suitable open set containing the positive real T axis.

In I we saw that

$$F_r = -\beta^{-1}\Big(\ln(2\cosh\beta E_1) + \int_0^\infty dE_2 P(E_2)\ln\cosh\beta E_2 + (4\pi)^{-1}\int_{-\pi}^{\pi} d\theta \ln|1+z_1 e^{i\theta}|^2 + (2\pi)^{-1} \times \int_{-\pi}^{0} d\theta \lim_{\mathfrak{M}\to\infty}\sum_{n=0}^{\mathfrak{M}-1}\ln\frac{C_{n+1}(\theta)}{C_n(\theta)}\Big) , \quad (3.2)$$

where $C_n(\theta)$ is determined from the recursion relation

$$\begin{bmatrix} C_{n+1}(\theta) \\ D_{n+1}(\theta)\end{bmatrix} = \begin{bmatrix} a^2+b^2 & a \\ a & 1\end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & z_2(n)^2\end{bmatrix}\begin{bmatrix} C_n(\theta) \\ D_n(\theta)\end{bmatrix} , \quad (3.3a)$$

with

$$C_0(\theta)=1, \quad D_0(\theta)=0 . \quad (3.3b)$$

Furthermore,

$$\lim_{\mathfrak{M}\to\infty}\mathfrak{M}^{-1}\sum_{n=0}^{\mathfrak{M}-1}\ln\frac{C_{n+1}(\theta)}{C_n(\theta)} = \lim_{\mathfrak{M}\to\infty}\mathfrak{M}^{-1}\sum_{n=0}^{\mathfrak{M}-1}\ln\left(a^2+b^2+\frac{a\lambda(n)D_n}{C_n}\right). \quad (3.4)$$

In this analysis, T may be made complex. The only change is that now a, b, and $z_2(n)$ will be complex and hence, in general, C_n and D_n will be complex. Therefore, if we define x_n as in I by

$$x_n = C_n/D_n , \quad (3.5)$$

then x_n will be a complex variable. However, once we make this generalization to complex x_n the argument of I goes through and we have

$$\nu(x) = \int d^2x' \int_0^\infty dE_2\, \delta^2\left(x - \frac{(a^2+b^2)x' + az_2^2}{ax'+z_2^2}\right) \times P(E_2)\nu(x') , \quad (3.6)$$

where

$$\delta^2(x-\bar{x}) = \delta[\mathrm{Re}(x-\bar{x})]\,\delta[\mathrm{Im}(x-\bar{x})] \quad (3.7)$$

and

$$d^2x = d\,\mathrm{Re}x\, d\mathrm{Im}x. \quad (3.8)$$

In (3. 6) the d^2x' integration is over the entire region of the x' plane where $\nu(x')$ does not vanish. In terms of $\nu(x)$ defined by (3. 6) we find that for complex T, F_r is given by (2. 1), except that the one-dimensional integral over dx is replaced by the two-dimensional integral over d^2x. Therefore, G will be an analytic function of T for T real if

$$\overline{G}(T) = \int_{-\pi}^{\epsilon} d\theta \int d^2x\, \nu(x) \int_0^\infty dE_2\, P(E_2) \times \ln(a^2+b^2+az_2^2x^{-1}) = \int_{-\pi}^{\epsilon} d\theta \int d^2x\, \nu(x) \int_0^\infty dE_2\, P(E_2)\ln(z_2^2+ax) \quad (3.9)$$

is an analytic function of T for T in an open set including the positive real T axis.

When T is real, $\nu(x)$ vanishes unless (2. 4) holds. When T is complex, however, $\nu(x)$ does not have the property that it is zero unless x is confined to some suitable one-dimensional subset of the x plane On the other hand, $\nu(x)$ does have the property that, when $a\neq 0$, it vanishes unless x is in some bounded region R of the complex x plane.

To determine this region we note that the homographic transformation appearing in (3. 6),

$$x = [(a^2+b^2)x' + az_2^2]/(ax'+z_2^2) , \quad (3.10)$$

may be rewritten as

$$\frac{x-x_0}{x+x_0^{-1}z_2^2} = B^2\frac{x'-x_0}{x'+x_0^{-1}z_2^2} , \quad (3.11)$$

where

$$B^2 = \frac{z_2^2(x_0-a)}{x_0(z_2^2+ax_0)} = \frac{a^2+b^2+z_2^2-[(a^2+b^2-z_2^2)^2+4a^2z_2^2]^{1/2}}{a^2+b^2+z_2^2+[(a^2+b^2-z_2^2)^2+4a^2z_2^2]^{1/2}} . \quad (3.12)$$

The points $x_0(z_2^2)$ and $-z_2^2x_0^{-1}(z_2^2)$ are the fixed points of the transformation (3. 10). Furthermore, if

$$a \geq \overline{\epsilon} > 0 , \quad (3.13)$$

then E_2 is real, and T is sufficiently close to the real axis

$$|B|^2 < 1 , \quad (3.14)$$

and it is possible to separate the curves $x_0(z_2^2)$ and $z_2^2x_0^{-1}(z_2^2)$ by a circle C that encloses $x_0(z_2^2)$ but not $z_2^2x_0^{-1}(z_2^2)$. As long as (3. 13) is satisfied, it is possible to find an open set S which contains the real positive T axis such that any transformation of the type (3. 10) will transform the circle C into its interior.[2] S clearly is a function of $\overline{\epsilon}$ and of C. Call the interior of such a separating circle R. Then, since $\nu(x)$ satisfies (3. 6), we conclude that if $a\neq 0$ and $T\in S$,

$$\nu(x) = 0 \quad \text{if} \quad x \notin R . \quad (3.15)$$

The region R so defined is clearly not the smallest region in which (3. 15) holds, but for the present purposes it is sufficient.

For each $\overline{\epsilon}\neq 0$ the set R is bounded. Therefore, we note with Furstenberg[3] that the solution of (3.6) may be written as

$$\nu(x)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\int_R K_k(x,y)\nu'(y)\,d^2y\ , \qquad (3.16)$$

where $\nu'(y)$ is some arbitrary function

$$K_1(x,y)=\int_0^\infty dE_2\,\delta^2\left(x-\frac{(a^2+b^2)y+az_2^2}{ay+z_2^2}\right)P(E_2)\ , \qquad (3.17)$$

$$K_k(x,y)=\int_R d^2\overline{y}\,K_{k-1}(x,\overline{y})K_1(\overline{y},y)\ , \qquad (3.18)$$

and, since $\nu(x)$ and $K_1(x,y)$ may contain δ functions, equality in (3.16) is meant in the sense that for a class of smooth test functions

$$\int_R d^2x\,fn\,(x)\nu(x)=\lim_{n\to\infty}\frac{1}{n}\int_R d^2x \times\int_R d^2y\ fn(x)\sum_{k=1}^{n}K_n(x,y)\nu'(y)\ . \qquad (3.19)$$

With this information on $\nu(x)$, we may investigate the analyticity of $\overline{G}$ defined by (3.9). For $\overline{G}$ to be analytic it is sufficient to prove that if $-\pi\leq\theta\leq\epsilon<0$ and $T\in S$, then

$$g(\theta,T)=\int_R d^2x\ \nu(x)\times\int_0^\infty dE_2\ P(E_2)\ \ln(a^2+b^2+az_2^2x^{-1}) \qquad (3.20)$$

is a continuous function of θ and T together and an analytic function of T for each θ.

We first consider $a\neq 0$ so that

$$-\pi<-\pi+\epsilon'\leq\theta\leq\epsilon<0\ . \qquad (3.21)$$

Then from (3.19) we may write

$$g(\theta,T)=\lim_{n\to\infty}\frac{1}{n}\int_R d^2x\int_R d^2y\sum_{k=1}^{n}K_k(x,y)\nu'(y)\times\int_0^\infty dE_2\ P(E_2)\ \ln(a^2+b^2+az_2^2x^{-1})\ . \qquad (3.22)$$

Consider the kth term in the sum

$$\int_R d^2x\int_R d^2y\int_0^\infty dE_2\ P(E_2)\ln(a^2+b^2+az_2^2x^{-1})\times K_k(x,y)\nu'(y)=\int_R d^2x\int_0^\infty dE_2^{(1)}dE_2^{(2)}\cdots dE_2^{(k)}\times P(E_2^{(1)})P(E_2^{(2)})\cdots P(E_2^{(k)})\times\int_0^\infty dE_2P(E_2)\ln(a^2+b^2+az_2^2x_k^{-1})\nu'(x), \qquad (3.23)$$

where x_k is the point that x transforms into when acted on by k transformations of the type (3.10) associated with $E_1^{(1)}\cdots E_2^{(k)}$. Now $\nu'(x)$ is an arbitrary function and hence may be chosen to be a function for which (3.17) is a continuous function of T and θ and an analytic function of T for each θ when (3.23) holds and $T\in S$. Furthermore,

$$\int_0^\infty dE_2P(E_2)\ln(a^2+b^2+az_2^2x^{-1})$$

is a function of x whose only singularities lie on the curve

$$x=-[a/(a^2+b^2)]z_2^2\quad(0\leq z_2<1) \qquad (3.24)$$

and at the (possibly isolated) point $x=0$. However, if $a\neq 0$ and $T\in S_\epsilon$, this curve does not intersect the set R. Therefore, we conclude that when (3.21) holds and $T\in S_\epsilon$, Eq. (3.23) is a continuous function of θ and T and an analytic function of T for each θ. But this argument is independent of k. Therefore, $g(\theta,T)$ possesses the same properties just shown to hold for (3.23) and hence

$$\int_{-\pi+\epsilon'}^{\epsilon}d\theta\ g(\theta,T)$$

is an analytic function of T.

To complete the proof that $\overline{G}(T)$ is an analytic function of T for $T\in S_\epsilon$ we must consider

$$-\pi\leq\theta\leq-\pi+\epsilon'\ . \qquad (3.25)$$

As $\theta\to-\pi$,

$$a(\theta)\to 0\ \text{ and }\ b(\theta)\to(1+z_1)/(1-z_1)\ . \qquad (3.26)$$

Therefore for all positive E_2

$$x_0(\lambda)\sim(b^2-\lambda)/a\ , \qquad (3.27)$$

and the previous argument will break down. However, if we rescale variables by defining

$$x_> = ax \qquad (3.28)$$

and

$$\nu_>(x_>)d^2x_> = \nu(x)d^2x\ , \qquad (3.29)$$

we find

$$\int_{-\pi}^{-\pi+\epsilon'}d\theta\int_R dx\ \nu(x)\int_0^\infty dE_2\ln(\lambda+ax)=\int_{-\pi}^{-\pi+\epsilon'}d\theta\times\int_{R_>}dx_>\,\nu_>(x_>)\int_0^\infty dE_2\ P(E_2)\ln[\lambda+x_>]\ , \qquad (3.30)$$

where $\nu_>(x_>)$ satisfies the equation

$$\nu_>(x_>)=\int_{R_>}d^2x'_>\int_0^\infty dE_2\ \delta^2\times\left(x_>-\frac{(a^2+b^2)x'_>+a^2z_2^2}{x'_>+z_2^2}\right)P(E_2)\nu(x'_>)\ . \qquad (3.31)$$

The transformation

$$x_> = [(a^2+b^2)x'_> + a^2 z_2^2]/(x'_> + z_2^2) , \tag{3.32}$$

which appears in (3.31) may be rewritten as

$$\frac{x_> - ax_0}{x_> + ax_0^{-1} z_2^2} = B^2 \frac{x'_> - ax_0}{x'_> + ax_0^{-1} z_2^2} . \tag{3.33}$$

When (3.25) holds, we conclude from (3.27) that the locus of fixed points $ax_0(z_2^2)$ and $az_2^2 x_0^{-1}(z_2^2)$ may be separated by a circle that encloses all points of the curve $ax_0(z_2^2)$, and this circle encloses a bounded region. Furthermore, because of (3.27), we see that (3.14) continues to hold for $T \in S$ even when $\theta = -\pi$. Therefore, the previous argument may be carried through to show that the region $R_>$ outside of which $\nu_>(x_>)$ vanishes is bounded when (3.25) holds and does not intersect the line

$$x_> = -z_2^2 . \tag{3.34}$$

Hence if $T \in S$, Eq. (3.30) is a continuous function of θ and T and an analytic function of T for fixed θ. Thus we have established that $\bar{G}$ (and hence G) is an analytic function of T for T real and non-negative.

IV. $\epsilon \leq \theta \leq 0$

When $\theta \to 0$, the arguments of Sec. III fail because, even if we make a scale change of variables like $x_> = ax$, it is not possible to separate the curves $x_0(z_2^2)$ and $z_2^2 x_0^{-1}(z_2^2)$ by a circle of finite radius that intersects neither curve when $\theta = 0$. However, there are types of restrictions that may be placed on $P(E_2)$ which will allow the previous argument to be made for $\theta = 0$. In particular, there are two cases for which we may therefore easily show that the free energy is an analytic function of T:

$$\mu(\lambda) \neq 0 \ \text{ only if } \lambda'_0 < \lambda < \lambda_0 < b^2(0) \tag{4.1a}$$

and

$$\mu(\lambda) \neq 0 \ \text{ only if } b^2(0) < \lambda'_0 < \lambda < \lambda_0 . \tag{4.1b}$$

In the first case, if all $E_2(j)$ were increased to the maximum allowed value E_2^0, the lattice would still be above T_c. In the second case, if all $E_2(j)$ were decreased to the minimum allowed value E'_2, the lattice would still be below T_c.

From (3.7) of I we see that for real T, $\nu(x)$ is nonzero only if

$$x_0(\lambda_0) < x < x_0(\lambda'_0) . \tag{4.2}$$

If (4.1a) holds, then $x_0(\lambda_0) \to \infty$ as $\theta \to 0$. Therefore, as in the case $\theta \to -\pi$, we may define $x_> = ax$ and the argument of Sec. III may be carried through word for word. Therefore if (4.1a) holds, F_r is an analytic function of T for T real and positive.

Similarly, if (4.1b) holds, then $x_0(\lambda'_0) \to 0$ as $\theta \to 0$. Accordingly, we define

$$x_< = x/a \tag{4.3a}$$

and

$$\nu_<(x_<)\, d^2x_< = \nu(x)\, d^2x . \tag{4.3b}$$

Then the contribution to F_r from $\epsilon \leq \theta \leq 0$ is

$$\int_\epsilon^0 d\theta \int_{R_<} d^2x_< \, \nu_<(x_<) \times \int_0^\infty dE_2 P(E_2) \ln(a^2+b^2+z_2^2/x_<) \tag{4.4}$$

when $R_<$ is the region in which $\nu_<(x_<)$ is nonzero. For T real, $R_<$ is the segment of the real axis

$$x_0(\lambda_0)/a \leq x_< \leq x_0(\lambda'_0)/a .$$

For T complex, even for $\theta \to 0$, the curve $x_0(\lambda)/a$ for $E'_2 < E_2^0$ may be separated from the curve $z_2^2 x_0^{-1}(\lambda)/a$ by a circle of finite radius which contains $x(\lambda)/a$. Therefore, we can apply the argument of Sec. III to conclude that if T is small enough so that (4.1b) holds, then F_r is an analytic function of T for T real and positive.

When (4.1) does not hold, this general argument cannot be made for the region $\theta \to 0$, and hence it is possible that this region may give a nonanalytic contribution to F_r.

When $\theta = 0$, the recursion relation (3.3) degenerates to

$$\begin{bmatrix} C_{n+1}(0) \\ D_{n+1}(0) \end{bmatrix} = \begin{bmatrix} b^2(0) & 0 \\ 0 & z_2^2(n) \end{bmatrix} \begin{bmatrix} C_n(0) \\ D_n(0) \end{bmatrix} . \tag{4.5}$$

Consequently, as noted in I, the associated stationary distributions for real T are either

$$\nu(x) = \delta(x) \tag{4.6a}$$

or

$$\nu(x) = (1/x^2)\delta(1/x) . \tag{4.6b}$$

The first expression gives the minimum contribution to the free energy if

$$\int_0^1 dE_2\, P(E_2) \ln(z_2^2/b^2) > 1 , \tag{4.7a}$$

while the second expression gives the minimum contribution to the free energy if

$$\int_0^1 dE_2\, P(E_2) \ln(z_2^2/b^2) < 1 . \tag{4.7b}$$

Accordingly, our strategy will be to find an approximate expression for $\nu(x)$ which reduces to (4.6) for $\theta = 0$. In studying such approximations we restrict T to be real and consider the analyticity properties of F_r, only after an explicit approximation to the nonanalytic part is obtained. This procedure will allow us to demonstrate that F_r is not analytic at T_c. Furthermore, as in I, it will also provide us with an approximation to the singularity at T_c. However, it is not possible to rigorously discuss the sense in which this leading approximation to the singularity at T_c is valid.

We first study the equation obtained from (2.7) by setting $a=0$ while keeping x fixed and of order 1. Then, if we call the resulting function $\nu_0(x)$, we have

$$\nu_0(x)=b^2x^{-2}\int_0^{xb^{-2}} dx'\ \nu_0(x')x'\mu\left[x'\left(\frac{b^2}{x}\right)\right] . \tag{4.8}$$

This equation has several solutions. First of all, it has the solutions given by (4.6). However, for our purposes it is necessary to realize that (4.8) has two other solutions, both of the form

$$\nu_0(x)=cx^{-1+\delta} , \tag{4.9}$$

where δ satisfies

$$1=\int_0^1 dt\ \ (t/b^2)^\delta \mu(t) . \tag{4.10}$$

Clearly $\delta=0$ satisfies (4.10), since by definition

$$\int_0^1 dt\ \mu(t)=1 . \tag{4.11}$$

However, since this solution is independent of T, it is useless for our purposes.

To see that (4.10) has one more solution consider the function

$$f(\delta)=\int_0^1 dt\ (t/b^2)^\delta \mu(t) . \tag{4.12}$$

Since (4.1) is assumed not to hold, it is clear that unless in (4.1) we let

$$\lambda_0=b^2 \ \text{ or } \ \lambda_0'=b^2 , \tag{4.13}$$

then as $\delta\to\pm\infty$

$$f(\delta)\to\infty . \tag{4.14}$$

[The cases where (4.13) hold may be studied by considering the $\delta\to\pm\infty$ limit of our final expression.] Furthermore,

$$f''(\delta)=\int_0^1 dt\ [\ln(t/b^2)]^2(t/b^2)^\delta\mu(t)>0 . \tag{4.15}$$

Combining (4.14) and (4.15) and the fact that $f(0)=1$, we see that if $f'(0)\neq 0$, there are two and only two values of δ which satisfy (4.10). However, if

$$\int_0^1 dt\ \ln(t/b^2)\mu(t)=f'(0)=0 , \tag{4.16}$$

the fixed solution at $\delta=0$ and the moving solution coincide.

Equation (4.16) is precisely Eq. (1.1), which determines T_c. Near T_c when $\delta\sim 0$ we may expand (4.10) to find that to leading order, δ is related to $T-T_c$ by

$$\delta\sim\frac{2\int_0^1 dt\ \mu(t)\ln(t/b^2)}{\int_0^1 dt\ \mu(t)[\ln(t/b^2)]^2} . \tag{4.17}$$

As an example, consider the case studied in I:

$$\begin{aligned}\mu(\lambda)&=N\lambda_0^{-N}\lambda^{N-1}, \quad 0\le\lambda\le\lambda_0\\ &=0 \qquad\quad , \ \text{otherwise}\end{aligned} \tag{4.18}$$

where N is large. Then

$$\int_0^1 dt\ \mu(t)\ln(t/b^2)=\ln(\lambda_0/b^2)-1/N \tag{4.19a}$$

and

$$\int_0^1 dt\ \mu(t)[\ln(t/b^2)]^2=[\ln(\lambda_0/b^2)]^2 - 2N^{-1}[\ln(\lambda_0/b^2)-1/N] . \tag{4.19b}$$

Therefore, $T=T_c$ when

$$\ln(\lambda_0/b^2)=N^{-1} , \tag{4.20}$$

and to leading order in $T-T_c$ and N, δ is given by

$$\delta\sim-2N^{-2}[\ln(b^2/\lambda_0)+N^{-1}]\sim 2N^{-2}(b^2\lambda_0^{-1}-1+N^{-1}) . \tag{4.21}$$

Combining (4.21) with (4.18) of I, we see that to leading order the δ of this paper and the δ of I are identical.

The function $\nu_0(x)$ is not integrable over the range 0 to ∞ and thus is not by itself a stationary distribution. However, when $T<T_c$, $\nu_0(x)$ fails to be integrable only at $x=0$. Therefore, for $T<T_c$, $\nu_0(x)$ will in some sense join on to the δ function (4.6a). Similarly, when $T>T_c$, $\nu_0(x)$ fails to be integrable only at $x=\infty$ and hence will in some sense join on to the δ function (4.6b).

To make these last remarks precise, we first notice that (2.7) reduces to (4.8) not only for $\theta\to 0$ and $x=0(1)$ but also for

$$aL<x<\epsilon'/a , \tag{4.22}$$

where

$$\epsilon'\ll 1 \ \text{ and } \ L\gg 1 . \tag{4.23}$$

Therefore, to consider in detail how $\nu_0(x)$ joins on to (4.6a) when $T<T_c$, we consider the region where $a\to 0$ and $x\to 0$ in such a fashion that $x_<$ is fixed. Then if we define

$$\nu_<(x_<)dx_<=\nu(x)dx \tag{4.24}$$

and let

$$\lim\nu_<(x_<)=\bar\nu_<(x_<) \text{ as } a\to 0, \tag{4.25}$$

we find that

$$\bar\nu_<(x_<)=\frac{b^2}{(x_<-1)^2}\int_{(1-b^2)^{-1}}^{x_<-1/b^2} dx_<'\,x_<'\,\bar\nu_<(x_<') \times\mu\left(x_<'\frac{b^2}{x_<-1}\right) \tag{4.26}$$

for

$$1/(1-b^2)\le x_<<\infty . \tag{4.27}$$

As $x_<\to\infty$, Eq. (4.26) is approximated as

$$\bar{\nu}_<(x_<) \sim \frac{b^2}{x_<^2} \int_{\bar{\epsilon} x_<}^{x_</b^2} dx'_< \, x'_< \, \bar{\nu}_<(x'_<) \mu\left(x'_< \frac{b^2}{x_<}\right)$$

$$= \frac{1}{b^2} \int_{\bar{\epsilon}}^1 dt \, t \bar{\nu}_<\left(\frac{t x_<}{b^2}\right) \mu(t) \,. \quad (4.28)$$

But $\bar{\epsilon}$ is arbitrarily small. Therefore, as $x_< \to \infty$,

$$\bar{\nu}_<(x_<) \to \text{const} \times \nu_0(x_<) = C_<(\delta) x^{-1+\delta} \,. \quad (4.29)$$

This is integrable at large $x_<$ for $T < T_c$. If $T < T_c$ ($\delta < 0$), we may normalize $\bar{\nu}_<(x_<)$ by

$$\int_{1/(1-b^2)}^{\infty} dx \, \bar{\nu}_<(x) = 1 \,. \quad (4.30a)$$

This integral will exist because of (4.29). However we will also need to consider $\bar{\nu}_<(x_<)$ when $T > T_c$. Therefore, at least if $\delta < 1$, it is convenient to normalize $\bar{\nu}_<(x_<)$ so that

$$\int_{1/(1-b^2)}^{\infty} dx_< [\bar{\nu}_<(x_<) - C_<(\delta) x_<^{-1+\delta}] - \delta^{-1} C_<(\delta)(1-b^2)^{-\delta} = 1 \,. \quad (4.30b)$$

Clearly, when $\delta < 0$, (4.30b) reduces to (4.30a).

From (4.25), (4.29), and (4.30a) we conclude that when $T < T_c$, $a \sim 0$ and $x/a = O(1)$

$$\nu(x) \sim (1/a) \bar{\nu}_<(x/a) \,. \quad (4.31)$$

We may use (4.31) in (2.1) to study the contribution to F_r from the region $\epsilon \le \theta \le 0$ when $T < T_c$ and (4.11) does not hold. This gives

$$\int_\epsilon^0 d\theta \int_{x_0(1)}^{(a^2+b^2)/a} dx \, \nu(x) \int_0^1 d\lambda \, \ln\left(a^2 + b^2 + \frac{\lambda a}{x}\right) \mu(\lambda)$$

$$\sim \int_\epsilon^0 d\theta \int_{x_0(1)}^{(a^2+b^2)/a} dx \, a^{-1} \bar{\nu}_<\left(\frac{x}{a}\right)$$

$$\times \int_0^1 d\lambda \ln\left(a + b^2 + \frac{\lambda a}{x}\right) \mu(\lambda)$$

$$\sim \int_\epsilon^0 d\theta \int_{1/(1-b^2)}^{\infty} dx_< \, \bar{\nu}(x_<) \int_0^1 d\lambda \ln(b^2 + \lambda x_<^{-1}) \mu(\lambda) \,. \quad (4.32)$$

This last integral is easily evaluated using the procedure of Sec. 2 of I. Indeed, because of (4.29),

$$\int_{1/(1-b^2)}^{\infty} dx_< \, \bar{\nu}_<(x_<) \int_0^1 d\lambda \ln(b^2 + \lambda x_<^{-1}) \mu(\lambda)$$

$$= \int_{1/(1-b^2)}^{\infty} dx_< \, \bar{\nu}_<(x_<) \int_0^1 d\lambda \ln\left(x_< \frac{b^2 + \lambda x_<^{-1}}{\lambda}\right)$$

$$\times \mu(\lambda) - \int_{1/(1-b^2)}^{\infty} dx_< \, \bar{\nu}_<(x_<) \ln x_< + \int_0^1 d\lambda \, \ln\lambda \, \mu(\lambda) \,. \quad (4.33)$$

In the first integral replace the variable λ by

$$q = x_<(b^2 + \lambda x_<^{-1})/\lambda \,, \quad (4.34)$$

to obtain

$$\int_{1/(1-b^2)}^{\infty} dq \int_{1/(1-b^2)}^{(q-1)/b^2} dx_< \frac{x_< b^2}{(q-1)^2} \bar{\nu}(x_<)$$

$$\times \ln q \mu\left(x_< \frac{b^2}{q-1}\right) - \int_{1/(1-b^2)}^{\infty} dx_< \, \bar{\nu}_<(x_<) \ln x_<$$

$$+ \int_0^1 d\lambda \, \mu(\lambda) \ln\lambda \,. \quad (4.35)$$

Then we may use (4.26) to show that the first two integrals cancel. Therefore (4.32) reduces to

$$\int_\epsilon^0 d\theta \int_0^1 d\lambda \, \mu(\lambda) \ln\lambda = -\epsilon \int_0^1 d\lambda \, \mu(\lambda) \ln\lambda \,, \quad (4.36)$$

an expression which is clearly an analytic function of T for T real and non-negative.

A similar argument may be used when $T > T_c$. Then if we consider $a \to 0$ and $x \to \infty$ such that $x_>$ is fixed, let

$$\nu_>(x_>) \, dx_> = \nu(x) \, dx \quad (4.37)$$

and define

$$\lim \nu_>(x_>) = \bar{\nu}_>(x_>) \quad \text{as } \theta \to 0 \,. \quad (4.38)$$

We find that (2.7) becomes

$$\bar{\nu}_>(x_>) = \frac{b^2}{x_>^2} \int_0^{\min[x_>/b^2 - x_>, \, b^2]} dx'_> \, x'_> \, \bar{\nu}_>(x'_>)$$

$$\times \mu\left(x'_> \frac{b^2 - x_>}{x_>}\right) \,. \quad (4.39)$$

As $x_> \to 0$, this approaches

$$\bar{\nu}_>(x_>) \sim \frac{b^2}{x_>^2} \int_0^{x_>/b^2} dx'_> \, x'_> \, \bar{\nu}_>(x'_>) \, \mu\left(x'_> \frac{b^2}{x_>}\right) \,. \quad (4.40)$$

This is the same equation as (4.8). Therefore, as $x_> \to 0$,

$$\bar{\nu}_>(x_>) \to \text{const} \times \nu_0(x_>) = C_>(\delta) x_>^{-1+\delta} \,, \quad (4.41)$$

which is integrable at $x_> = 0$ if $T > T_c$. Therefore, if $T > T_c$, $a \sim 0$, and $xa = O(1)$,

$$\nu(x) \sim a \bar{\nu}_>(xa) \,, \quad (4.42)$$

where, if $T > T_c$ ($\delta > 0$), we normalize $\bar{\nu}_>(x_>)$ by the requirement

$$\int_0^{b^2} dx_> \ \bar{\nu}_>(x_>) = 1 \qquad (4.43a)$$

and if $\delta > -1$, we use the condition

$$\int_0^{b^2} dx_> \ [\bar{\nu}_>(x_>) - C_>(\delta) x^{-1+\delta}] + \delta^{-1} C_>(\delta) b^{2\delta} = 1 , \qquad (4.43b)$$

which reduces to (4.43a) if $\delta > 0$. Approximation (4.42) gives a contribution to F_r from $\epsilon \le \theta \le 0$ of

$$\int_\epsilon^0 d\theta \int_{x_0(1)}^{(a^2+b^2)/a} dx \ \nu(x) \int_0^1 d\lambda \ \mu(\lambda) \ln(\lambda + ax)$$

$$\sim \int_\epsilon^0 d\theta \int_0^{b^2} dx_> \ \bar{\nu}_>(x_>) \times \int_0^1 d\lambda \ \mu(\lambda) \ln(\lambda + x) . \qquad (4.44)$$

Because (4.43a) holds if $T > T_c$, we may proceed as we did for (4.33) to evaluate this expression as

$$\int_\epsilon^0 d\theta \ln b^2 = -\epsilon \ln b^2 , \qquad (4.45)$$

which, again, is clearly an analytic function of T for T real and non-negative.

It remains for us to investigate the case $T = T_c$. The previous arguments fail because if $T = T_c$, then $\delta = 0$ and hence $\bar{\nu}_>(x_>)$ fails to be integrable at $x_> \to 0$. In this case, when $\theta \sim 0$, the dominant contribution to the normalization factor is coming from the range $x = O(1)$, where $\nu(x)$ is given by const $\times\, x^{-1}$. When $x \sim a$, $\nu(x)$ behaves as $\bar{\nu}_<(x/a)$, which is integrable at $x \to 0$. Similarly, when $x \sim 1/a$, $\nu(x)$ behaves as $\bar{\nu}_>(ax)$, which is integrable as $x \to \infty$. Therefore, when $< a < x < \epsilon^1/a$ and $\delta \sim 0$, we may approximate

$$\nu(x) \sim \text{const} \times \frac{x^{-1+\delta}}{(\epsilon'/a)^\delta - (La)^\delta} . \qquad (4.46)$$

This is normalized to some fixed constant. As $\delta \to 0$, it becomes

$$\nu(x) \sim \text{const} \times \frac{x^{-1}}{\ln(\epsilon'/La^2)} . \qquad (4.47)$$

For our purposes it is useful to have an approximate expression for $\nu(x)$ which, as $a \to 0$, reduces to (4.46) if $x = O(1)$ and $\delta \sim 0$, while it reduces to $\bar{\nu}_>$ if $T > T_c$, $x = O(a)$ and to $\bar{\nu}_>$ if $T > T_c$ and $x = O(a^{-1})$. Such an approximation is

$$\bar{\nu}(x) = \frac{C_>(\delta) a^{\delta-1} \bar{\nu}_<(x/a)}{C_<(\delta) a^{-\delta} + C_>(\delta) a^\delta} \quad \text{if } x < 1$$

$$= \frac{C_<(\delta) a^{-\delta+1} \nu_>(xa)}{C_<(\delta) a^{-\delta} + C_>(\delta) a^\delta} \quad \text{if } x > 1 . \qquad (4.48)$$

Clearly, as $a \to 0$, if $\delta < 0$ and $x = O(a)$, this reduces to $a^{-1}\nu_<(x_<)$, while if $\delta > 0$ and $x = O(a)$, it reduces to $a\nu_>(x_>)$. Furthermore, as $a \to 0$, $\nu(x)$ is normalized to 1 since

$$\int_{x_0(1)}^{(a^2+b^2)/a} dx \ \bar{\nu}(x) = \frac{C_>(\delta) a^\delta}{C_<(\delta) a^{-\delta} + C_>(\delta) a^\delta} \int_{x_0(1)/a}^{a^{-1}} dx_< \ \bar{\nu}_<(x_<) + \frac{C_<(\delta) a^{-\delta}}{C_<(\delta) a^{-\delta} + C_>(\delta) a^\delta} \int_a^{a^2+b^2} dx_> \ \bar{\nu}_>(x_>)$$

$$= [C_<(\delta) a^{-\delta} + C_>(\delta) a^\delta]^{-1} \Big[C_>(\delta) a^\delta \Big(\int_{x_0(1)/a}^{a^{-1}} dx_< [\bar{\nu}_<(x_<) - C_<(\delta) x^{-1+\delta}] + \delta^{-1} C_<(\delta) \{a^{-\delta} - [x_0(1) a^{-1}]^\delta\} \Big)$$

$$+ C_<(\delta) a^{-\delta} \Big(\int_a^{a^2+b^2} dx_> \ [\bar{\nu}_>(x_>) - C_>(\delta) x_>^{-1+\delta}] + \delta^{-1} C_>(\delta) [(a^2+b^2)^\delta - a^\delta] \Big) \Big] , \qquad (4.49)$$

which, using (4.30b) and (4.43b), reduces to $1 + O(a)$ (at least if $-1 < \delta < 1$). Finally, we see from (4.30b) that

$$\lim \delta^{-1} C_<(\delta) = -1 \quad \text{as } \delta \to 0 \qquad (4.50a)$$

and from (4.43b) that

$$\lim \delta^{-1} C_>(\delta) = 1 \text{ as } \delta \to 0 . \qquad (4.50b)$$

Therefore, for $\delta = 0$, $a \sim 0$, and $x = O(1)$, (4.48) reduces to

$$x^{-1}/(2 \ln a) \qquad (4.51)$$

as desired.

We now may study the analyticity properties of F_r at $T = T_c$ by using (4.48) in (2.1) when $\theta \sim 0$, to obtain

$$\int_\epsilon^0 d\theta \, [C_<(\delta) a^{-\delta} + C_>(\delta) a^\delta]^{-1} \Big[\int_{x_0(1)}^1 dx \, C_>(\delta) a^{-1+\delta} \bar{\nu}_<(x/a) \int_0^1 d\lambda \, \mu(\lambda) \ln(a^2 + b^2 + a\lambda/x)$$

$$+ \int_1^{a^2+b^2/a} dx \, C_<(\delta) a^{-\delta+1} \bar{\nu}_>(xa) \int_0^1 dx \, \mu(\lambda) \ln(a^2 + b^2 + a\lambda/x) \Big] . \qquad (4.52)$$

Consider first $-1<\delta<0$. Then (4.52) is conveniently rewritten as

$$\int_\epsilon^0 d\theta[C_<(\delta)a^{-\delta}+C_>(\delta)a^{\delta}]^{-1}\{\int_{x_0(1)}^{\infty}dx\,C_>(\delta)a^{\delta-1}\overline{\nu}_<(x/a)\int_0^1 d\lambda\,\mu(\lambda)\ln(a^2+b^2+a\lambda/x)$$
$$+\int_1^{(a^2+b^2)/a}dx[C_<(\delta)a^{-\delta+1}\overline{\nu}_>(xa)-C_>(\delta)a^{\delta-1}\overline{\nu}_<(x/a)]\int_0^1 d\lambda\,\mu(\lambda)\ln(a^2+b^2+a\lambda/x)$$
$$-\int_{(a^2+b^2)/a}^{\infty}dx\,C_>(\delta)a^{\delta-1}\overline{\nu}_<(x/a)\int_0^1 d\lambda\,\mu(\lambda)\ln(a^2+b^2+a\lambda/x)\}. \quad (4.53)$$

Therefore, using (4.29), we find [correct to terms of $O(a)$] that the integrand of the θ integral in (4.53) is

$$[C_<(\delta)a^{-\delta}+C_>(\delta)a^{\delta}]^{-1}\{C_>(\delta)a^{\delta}\int_{1/(1-b^2)}^{\infty}dx_<\overline{\nu}_<(x_<)\int_0^1 d\lambda\,\mu(\lambda)\ln(b^2+\lambda/x)$$
$$+C_<(\delta)a^{-\delta}\ln b^2[\int_0^{b^2}dx_>(\overline{\nu}_>(x_>)-C_>(\delta)x^{-1+\delta})+\delta^{-1}C_>(\delta)b^{2\delta}]\}, \quad (4.54)$$

which, using (4.32), (4.36), and (4.43b), reduces to

$$\frac{C_<(\delta)a^{-\delta}\ln b^2+C_>(\delta)a^{\delta}\int_0^1 d\lambda\,\mu(\lambda)\ln\lambda}{C_<(\delta)a^{-\delta}+C_>(\delta)a^{\delta}}. \quad (4.55)$$

A similar procedure may be used when $0<\delta<1$ to show that the integrand of the θ integration of (4.52) is also given by (4.55). Therefore we conclude that when $\delta\sim0$, the most singular part of F_r is given by

$$\int_\epsilon^0 d\theta\,\frac{C_<(\delta)a^{-\delta}\ln b^2+C_>(\delta)a^{\delta}\int_0^1 d\lambda\,\mu(\lambda)\ln\lambda}{C_<(\delta)a^{-\delta}+C_>(\delta)a^{\delta}}. \quad (4.56)$$

It remains to make the singularity in (4.56) at $\delta=0$ explicit. We consider only $\delta\geq0$, since the case $\delta\leq0$ is treated in an identical manner. To do this we first rewrite (4.56) as

$$-\epsilon-\int_0^1 d\lambda\,\mu(\lambda)\ln\left(\frac{\lambda}{b^2}\right)\int_\epsilon^0 d\theta\left(1+\frac{C_>(\delta)}{C_<(\delta)}\,a^{2\delta}\right)^{-1}. \quad (4.57)$$

But near $\delta=0$ we see from (4.17) that

$\int_0^1 d\lambda\,\mu(\lambda)\ln(\lambda/b^2)$

is proportional to δ and from (4.50) that $C_>(\delta)/C_<(\delta)\sim-1$. Therefore the dominant singularity in (4.57) at $\delta=0$ is the same as that in

$$2\delta\int_0^1 d\theta\,(1-\theta^{2\delta})^{-1}. \quad (4.58)$$

But this is precisely the integral [(4.44) of I] which gave the leading singularity in F_r for the special case (1.3). Accordingly, we conclude, as in I, that at $\delta=0$ the most singular part of F_r has the formal expansion

$$\tfrac{1}{2}\delta+\sum_{n=0}^{\infty}B_{2n}\,\delta^{2n}(2n)^{-1}, \quad (4.59)$$

where B_{2n} are the Bernoulli numbers. Therefore, for any distribution function $P(E_2)$ except $\delta(E_2-E_2^0)$, the most singular part of F_r is an infinitely differentiable function of the real variable T even at $T=T_c$. However, (4.59) diverges for all $\delta\neq0$, and hence we conclude that for any $P(E_2)$, F_r fails to be an analytic function of T at $T=T_c$, where T_c is given by (1.1).

ACKNOWLEDGMENT

We would like to thank Professor T. T. Wu for many illuminating discussions.

*Work supported in part by U. S. Atomic Energy Commission Contract No. AT-(30-1)-3668B.

[1]B. M. McCoy and T. T. Wu, Phys. Rev. 176, 631 (1968). This paper will henceforth be referred to as I.

[2]L. R. Ford, *Automorphic Functions* (Chelsea, New York, 1951), Chaps. 1 and 2.

[3]H. Furstenberg, Trans. Am. Math. Soc. 108, 377 (1963).